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The distribution of zeros of general q-polynomials

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Abstract. A general system of q-orthogonal polynomials is defined by means of its three-term recurrence relation. This system encompasses many of the known families of q-polynomials, among them the q-analogue of the classical orthogonal polynomials. The asymptotic density of zeros of the system is shown to be a simple and compact expression of the parameters which characterize the asymptotic behaviour of the coefficients of the recurrence relation. This result is applied to specific classes of polynomials known by the names q-Hahn, q-Kravchuk, q-Racah, q-Askey and Wilson, Al Salam–Carlitz and the celebrated little and big q-Jacobi.

1. Introduction

In the last decade there has been increasing interest in the so-called q-orthogonal polynomials (or basic orthogonal polynomials; for a review see [1–3]). The reason is not only of a purely intrinsic nature but also because of the many applications in several areas of mathematics (e.g. continued fractions, Eulerian series, theta functions, elliptic functions, etc; see for instance [4, 5]) and physics (e.g. angular momentum [6, 7] and its q-analogue [8–11], the q-Schrödinger equation [12] and q-harmonic oscillators [13–19]). Moreover, it is well known that the connection between the representation theory of quantum algebras (Clebsch–Gordan coefficients, $3j$ and $6j$ symbols) and the q-orthogonal polynomials (see [20, 21, vol III, 22–24]), and the important role that these q-algebras play in physical applications (see for instance [26–31] and references therein).

However, the distribution of zeros of these polynomials remains practically unknown to the best of our information. The present paper continues, corrects and considerably extends the investigation of the asymptotic behaviour of zeros of the q-polynomials initiated by Dehesa [32]. This is done by the consideration of a general system of q-polynomials which includes most of the q-polynomials encountered in the literature and the study of its distribution density of zeros as well as the corresponding asymptotic limit.

The method of proof used is very straightforward; it is based on an explicit formula for the moments-around-the-origin of the discrete density of zeros of a polynomial with a given degree in terms of the coefficients of the three-term recurrence relation [37], as described in lemma 1 below. This method was previously applied to normal (non-q) polynomials where

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recurrence coefficients are given by means of a rational function of degree [38], as well as to corresponding Jacobi matrices [39] encountered in the quantum mechanical description of some physical systems.

The paper is structured as follows. First, in section 2, a general set of q-polynomials $\{P_n(x)_q\}_{n=0}^N$ are introduced by means of its three-term recurrence relation. Section 3 contains the main results which refer to the discrete density of zeros (i.e. the number of zeros per unit of zero interval) of the polynomial $P_n(x)_q$, n being a sufficiently large value, and to its asymptotical limit (i.e. when $n \to \infty$). Both discrete and asymptotic densities of zeros are supposed to be characterized by the knowledge of all their moments. These results are given in the form of four theorems. Theorem 1 gives the behaviour of the moments of the discrete density of zeros in terms of the parameters defining the recurrence relation. The asymptotic density of zeros is given by theorems 2–4 in a similar way.

Proofs and a detailed discussion of these theorems are contained in sections 4 and 5 respectively. The utmost effort has been concentrated on searching for an appropriate asymptotic density of zeros to obtain as much information as possible about the asymptotic distribution of zeros of the new polynomials. Finally, section 6 contains applications of theorems 1–4 formulated in section 3 to several known families of q-polynomials.

2. The general system of q-orthogonal polynomials

The general system of q-orthogonal polynomials $\{P_n(x)_q\}_{n=0}^N$ is defined by the recurrence relation

$$\begin{aligned} &P_n(x) = (x - a_n)P_{n-1}(x) - b_{n-1}^2 P_{n-2}(x) \\ &P_{-1}(x) = 0 \qquad P_0(x) = 1 \qquad n \geqslant 1 \end{aligned} \tag{1}$$

with the coefficients a_n and b_{n-1}^2 given by

$$\begin{aligned} a_n &= \frac{\sum_{m=0}^{A} \left(\sum_{i=0}^{g_m} \alpha_i^{(m)} n^{g_m - i}\right) q^{d_m n}}{\sum_{m=0}^{A'} \left(\sum_{i=0}^{h_m} \beta_i^{(m)} n^{h_m - i}\right) q^{e_m n}} \equiv \frac{a_n^{\text{num}}}{a_n^{\text{den}}} \\ b_n^2 &= \frac{\sum_{m=0}^{B} \left(\sum_{i=0}^{k_m} \theta_i^{(m)} n^{k_m - i}\right) q^{f_m n}}{\sum_{m=0}^{B'} \left(\sum_{i=0}^{l_m} \gamma_i^{(m)} n^{l_m - i}\right) q^{s_m n}} \equiv \frac{(b_n^{\text{num}})^2}{(b_n^{\text{den}})^2} \end{aligned} \tag{2}$$

where q is an arbitrary positive real number bigger than 1. Further, the following general requirements on the real parameters defining a_n and b_n^2 will be assumed.

(1) All members of the sequence $\{\beta_i^{(m)}; 0 \leqslant i \leqslant h_m\}_{m=0}^{A'}$, $\{\gamma_i^{(m)}; 0 \leqslant i \leqslant l_m\}_{m=0}^{B'}$ do not vanish simultaneously. So a_n and b_n^2 not to be infinite for all n is assured.

(2) The parameters $\{\theta_i^{(m)}; 0 \leqslant i \leqslant k_m\}_{m=0}^{B}$ and $\{\gamma_i^{(m)}; 0 \leqslant i \leqslant l_m\}_{m=0}^{B'}$ are such that $b_n^2 > 0$ for $n \geqslant 1$. Then Favard's theorem assures the orthogonality of the polynomials $\{P_n(x)_q\}_{n=0}^N$.

(3) The following inequalities are verified:

$$\begin{aligned} &q^{d_0} > q^{d_1} > \cdots > q^{d_A} \qquad q^{e_0} > q^{e_1} > \cdots > q^{e_{A'}} \\ &q^{f_0} > q^{f_1} > \cdots > q^{f_B} \qquad q^{s_0} > q^{s_1} > \cdots > q^{s_{B'}} \end{aligned} \tag{3}$$

and

$$\begin{aligned} &g_0 > g_1 > \cdots > g_m \qquad h_0 > h_1 > \cdots > h_m \\ &k_0 > k_1 > \cdots > k_m \qquad l_0 > l_1 > \cdots > l_m \end{aligned}. \tag{4}$$

Conditions (3) and (4) do not obviously imply any loss of generality. Here it should also be pointed out that the polynomials discussed in [32] are instances of the polynomials (1), (2) corresponding to the values $g_m = k_m = h_m = e_m = l_m = s_m = 0$ for all m.

3. Main results

Before collecting the main results of this work, let us describe lemma 1 which is the basic tool to find them.

Lemma 1. Let $\{P_N(x)\}$ be a system of polynomials $P_N(x)$ defined by the recurrence relation (1), which is characterized by the sequences of numbers $\{a_n\}$ and $\{b_n\}$. Let the quantities

$$\mu_0 = N \qquad \mu_m'^{(N)} = \int_a^b x^m \rho_N(x)\,\mathrm{d}x \qquad m = 1, 2, \ldots, N \tag{5}$$

be the non-normalized-to-unity spectral moments of the polynomials $P_N(x)$, i.e. the moments around the origin of the discrete density of zeros $\rho_N(x)$, defined by

$$\rho_N(x) = \sum_{i=1}^{N} \delta(x - x_{N,i}) \tag{6}$$

$\{x_{N,i}, i = 1, 2, \ldots, N\}$ being the zeros of that polynomial. It is fulfilled that

$$\mu_m'^{(N)} = \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \sum_{i=1}^{N-t} a_i^{r_1'} b_i^{2r_1} a_{i+1}^{r_2'} b_{i+1}^{2r_2} \ldots b_{i+j-1}^{2r_j} a_{i+j}^{r_{j+1}'} \tag{7}$$

for $m = 1, 2, \ldots, N$. The summation $\sum_{(m)}$ runs over all partitions $(r_1', r_1, \ldots, r_{j+1}')$ of the number m such that

(1) $R' + 2R = m$, where R and R' denote the sums $R = \sum_{i=1}^{j} r_i$ and by $R' = \sum_{i=1}^{j-1} r_i'$, or

$$\sum_{i=1}^{j-1} r_i' + 2\sum_{i=1}^{j} r_i = m \tag{8}$$

(2) if $r_s = 0$, $1 < s < j$, then $r_k = r_k' = 0$ for each $k > s$ and

(3) $j = \frac{m}{2}$ or $j = \frac{m-1}{2}$ for m even or odd respectively.

The factorial coefficients F are defined by

$$F(r_1', r_1, r_2', \ldots, r_{p-1}', r_{p-1} r_p') = m\frac{(r_1' + r_1 - 1)!}{r_1'! r_1!}\left[\prod_{i=2}^{p-1} \frac{(r_{i-1} + r_i' + r_i - 1)!}{(r_{i-1} - 1)! r_i! r_i'!}\right] \times \frac{(r_{p-1} + r_p' - 1)!}{(r_{p-1} - 1)! r_p'!} \tag{9}$$

with the convention $r_0 = r_p = 1$. For the evaluation of these coefficients, we must take into account the following convention

$$F(r_1', r_1, r_2', r_2 \ldots, r_{p-1}', 0, 0) = F(r_1', r_1, r_2', r_2 \ldots, r_{p-1}').$$

In (7), t denotes the number of non-vanishing r_i which are involved in each partition of m.

This lemma was initially found in a context of Jacobi matrices [37, 38]. Just to understand the practical use of the lemma, let us give the first three spectral moments

$$\begin{aligned} \mu_1' &= \sum_{i=1}^{N} a_i \\ \mu_2' &= \sum_{i=1}^{N} a_i^2 + 2\sum_{i=1}^{N-1} b_i^2 \\ \mu_3' &= \sum_{i=1}^{N} a_i^3 + 3\sum_{i=1}^{N-1} b_i^2 (a_i + a_{i+1}). \end{aligned} \tag{10}$$

In the following, the main results of this work are collected in the form of four theorems. The first of them refers to the discrete density of zeros (6) of the polynomials defined by (1), (2) and the other three are concerned with the asymptotic density of zeros, i.e. when the degree of the polynomial tends towards infinity. Throughout the paper the symbol $\sim$ means *behaves as*.

Theorem 1. Let $P_N(x)_q$, very large N, be a polynomial defined by the expressions (1)–(4). The moments $\{\mu_m'^{(N)}; m = 1, 2, \ldots, N\}$ of the non-normalized density of zeros $\rho_N(x) = \sum_{i=1}^{N} \delta(x - x_{N,i})$ of the polynomial $P_N(x)_q$ have the following behaviour

(1) If $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0$, three cases occur.

(a) If $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m N^{(g_0-h_0)m+1}. \tag{11}$$

(b) If $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{\frac{1}{2}(k_0-l_0)m+1}. \tag{12}$$

(c) If $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} N^{\frac{1}{2}(k_0-l_0)m+1}. \tag{13}$$

(2) If $d_0 - e_0 \neq 0$ and/or $f_0 - s_0 \neq 0$, two cases occur.

(a) (i) If $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$ in such a way that $\Omega_1 \neq 0$, then

$$\mu_m'^{(N)} \sim \sum_{(m)} \frac{F(r_1', r_1, \ldots, r_{j+1}')}{q^{-\Omega_2}(\ln q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M} \left(\frac{q^{\Omega_1}}{1 - q^{\Omega_1}}\right) \tag{14}$$

where $\frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M}$ denotes the M derivative with respect to Ω_1.

(ii) If $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$ and $g_0 - h_0 = k_0 - l_0 = 0$, then

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', 0, \ldots, 0, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} N. \tag{15}$$

(iii) If $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$ and $g_0 - h_0 = k_0 - l_0 = 0$, then

$$\mu_m'^{(N)} \sim \sum_{(m)} F(0, r_1, \ldots, r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N. \tag{16}$$

(b) If $d_0 - e_0 > 0$ and/or $f_0 - s_0 > 0$, three different subcases may occur, namely:

(i) $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} N^{(g_0-h_0)m}. \tag{17}$$

(ii) If $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$, then three different types still come up.

(A) If $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} N^{(g_0-h_0)m}. \tag{18}$$

(B) If $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{q^{\Omega_2 + m(N+1-t)(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} N^{m(g_0-h_0)}. \tag{19}$$

(C) If $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{(d_0-e_0)mN}}{q^{(d_0-e_0)m} - 1} N^{\frac{1}{2}(k_0-l_0)m}. \tag{20}$$

(iii) $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$, then

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{\frac{1}{2}(f_0-s_0)mN}}{q^{\frac{1}{2}(f_0-s_0)m} - 1} N^{\frac{1}{2}(f_0-s_0)m}. \tag{21}$$

The summation $\sum_{(m)}$ and the parameter t are as defined in lemma 1. Besides, the parameters Ω_1, Ω_2 and M are as follows:

$$\Omega_1 = \left[(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)\right] R' + \frac{m}{2}(f_0 - s_0) \tag{22}$$

$$\Omega_2 = (d_0 - e_0) \sum_{k=1}^{j} k r_{k+1}' + 2(f_0 - s_0) \sum_{k=1}^{j-1} k r_{k+1} \tag{23}$$

$$M = \left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right] R' + \frac{m}{2}(k_0 - l_0). \tag{24}$$

The proof of this theorem is shown in section 4.

Theorem 2. Let $P_N(x)_q$ be a polynomial defined as in theorem 1 with the additional condition $(d_0 - e_0) = \frac{1}{2}(f_0 - s_0) = 0$ (i.e. case 1).

Let $\rho(x)$, $\rho_1^*(x)$ and $\rho_2^*(x)$ be the asymptotic (i.e. when $N \to \infty$) densities of zeros of the polynomial $P_N(x)_q$ defined by

$$\begin{aligned} \rho(x) &= \lim_{N\to\infty} \rho_N(x) \\ \rho_1^*(x) &= \lim_{N\to\infty} \frac{1}{N} \rho_N\left(\frac{x}{N^{(g_0-h_0)}}\right) \\ \rho_2^*(x) &= \lim_{N\to\infty} \frac{1}{N} \rho_N\left(\frac{x}{N^{\frac{1}{2}(k_0-l_0)}}\right) \end{aligned} \tag{25}$$

and their corresponding moments are as follows:

$$\begin{aligned} \mu_m' &= \lim_{N\to\infty} \mu_m'^{(N)} \\ \mu_m^*(1) &= \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(g_0-h_0)m}} \\ \mu_m^*(2) &= \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(k_0-l_0)\frac{m}{2}}} \end{aligned} \tag{26}$$

for $m = 0, 1, 2, \ldots$, respectively. here $\rho_N(x)$ denotes the (discrete) density of zeros of the polynomial $P_N(x)_q$. It turns out that

$$\mu_m' = \infty \qquad m \geqslant 0 \tag{27}$$

and

(1) if $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m^*(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \qquad m \geqslant 0 \tag{28}$$

(2) if $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m^*(2) = \sum_{(m)} F(r_1', r_1, \ldots, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \qquad m \geqslant 0 \tag{29}$$

(3) if $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m^*(2) = \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \qquad m \geqslant 0. \tag{30}$$

Here the coefficients F and the symbol of summation $\sum_{(m)}$ are as in theorem 1.

Theorem 3. Let $P_N(x)_q$ be a polynomial defined as in theorem 1 with the additional condition $(d_0 - e_0) \leqslant 0$ and $\frac{1}{2}(f_0 - s_0) \leqslant 0$ (i.e. case 2a).

Let $\rho(x)$ and $\rho_1(x)$ be the asymptotic densities of zeros of the polynomial $P_N(x)_q$ defined by

$$\rho(x) = \lim_{N\to\infty} \rho_N(x) \qquad \rho_1(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N(x) \tag{31}$$

and their corresponding moments are as follows:

$$\mu_m' = \lim_{N\to\infty} \mu_m'^{N} \qquad \mu_m'(1) = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N} \tag{32}$$

for $m \geqslant 0$, respectively. It turns out that:

(1) If $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$ in such a way that $\Omega_1 \neq 0$, then

$$\mu_m' = \sum_{(m)} \frac{F(r_1', r_1, \ldots, r_{j+1}')}{q^{-\Omega_2}(\ln q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M}\left(\frac{q^{\Omega_1}}{1 - q^{\Omega_1}}\right) \tag{33}$$

and

$$\mu_0'(1) = 1 \qquad \mu_m'(1) = 0 \qquad m \geqslant 1. \tag{34}$$

(2) If $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$ and $g_0 - h_0 = k_0 - l_0 = 0$, then

$$\mu_m' = \infty \qquad m \geqslant 0 \tag{35}$$

$$\mu_m'(1) = \begin{cases} 1 & m = 0 \\ \displaystyle\sum_{(m)} F(r_1', 0, \ldots, 0, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} & m \geqslant 1. \end{cases} \tag{36}$$

(3) If $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$ and $g_0 - h_0 = k_0 - l_0 = 0$, then

$$\mu_m' = \infty \qquad m \geqslant 0 \tag{37}$$

$$\mu_m'(1) = \begin{cases} 1 & m = 0 \\ \displaystyle\sum_{(m)} F(0, r_1, 0, \ldots, r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} & m \geqslant 1. \end{cases} \tag{38}$$

Here the coefficients F and the symbol of summation $\sum_{(m)}$ and the parameters Ω_1, Ω_2 and M are as in theorem 1.

Theorem 4. Let $P_N(x)_q$ be a polynomial defined as in theorem 1 with the additional condition $(d_0 - e_0) > 0$ and/or $\frac{1}{2}(f_0 - s_0) > 0$ (i.e. case 2b).

Let $\rho(x)$, $\rho_1^{**}(x)$, $\rho_2^{**}(x)$, $\rho_3^{**}(x)$, $\rho_1^{++}(x)$, $\rho_2^{++}(x)$ and $\rho_3^{++}(x)$ be the asymptotic densities of zeros of the polynomial $P_N(x)_q$ given by

$$\rho(x) = \lim_{N\to\infty} \rho_N(x) \tag{39}$$

$$\begin{aligned}
\rho_1^{**}(x) &= \lim_{N\to\infty} \rho_N\left(\frac{xq^{-(d_0-e_0)N}}{N^{(g_0-h_0)}}\right) \\
\rho_2^{**}(x) &= \lim_{N\to\infty} \rho_N\left(\frac{xq^{-(d_0-e_0)N}}{N^{\frac{1}{2}(k_0-l_0)}}\right) \\
\rho_3^{**}(x) &= \lim_{N\to\infty} \rho_N\left(\frac{xq^{-\frac{1}{2}(f_0-s_0)N}}{N^{\frac{1}{2}(k_0-l_0)}}\right)
\end{aligned} \tag{40}$$

$$\begin{aligned}
\rho_1^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N\left(\frac{xq^{-(d_0-e_0-1)N}}{N^{(g_0-h_0)}}\right) \\
\rho_2^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N\left(\frac{xq^{-(d_0-e_0-1)N}}{N^{\frac{1}{2}(k_0-l_0)}}\right) \\
\rho_3^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N\left(\frac{xq^{-\frac{1}{2}(f_0-s_0-2)N}}{N^{\frac{1}{2}(k_0-l_0)}}\right)
\end{aligned} \tag{41}$$

and their corresponding moments are as follows:

$$\mu'_m = \lim_{N\to\infty} \mu'^{(N)}_m \tag{42}$$

$$\begin{aligned}
\mu_m^{**}(1) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{(g_0-h_0)} q^{(d_0-e_0)mN}} \\
\mu_m^{**}(2) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{(d_0-e_0)mN}} \\
\mu_m^{**}(3) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{\frac{1}{2}(f_0-s_0)mN}}
\end{aligned} \tag{43}$$

$$\begin{aligned}
\mu_m^{++}(1) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{(g_0-h_0)} q^{(d_0-e_0-1)mN}} \\
\mu_m^{++}(2) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{(d_0-e_0-1)mN}} \\
\mu_m^{++}(3) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{\frac{1}{2}(f_0-s_0-2)mN}}
\end{aligned} \tag{44}$$

for $m \geqslant 0$, respectively, and where symbol $(n)_q$ denotes the q-basic number

$$(n)_q = \frac{q^n - 1}{q - 1} \tag{45}$$

related with the q-numbers $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ by formula $(n)_q = q^{\frac{n-1}{2}}[n]_{q^{\frac{1}{2}}}$. It turns out that

$$\mu'_m = \infty \qquad m \geqslant 0 \tag{46}$$

and

(1) $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$, then

$$\mu_m^{**}(1) = \begin{cases} \infty & m = 0 \\ \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} & m \geqslant 1. \end{cases} \tag{47}$$

Also,

$$\mu_m^{++}(1) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu_m^{**}(1) & m \geqslant 1. \end{cases} \tag{48}$$

(2) If $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$, then three different situations arise.

(a) If $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$. Then the moments $\mu_m^{**}(1)$ and $\mu_m^{++}(1)$ have the same values as in the previous case, i.e. as formulae (47), (48).

(b) If $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m^{**}(1) = \begin{cases} \infty & m = 0 \\ \sum_{(m)} F(r_1', r_1, \ldots, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{q^{\Omega_2 + m(1-t)(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} & m \geqslant 1. \end{cases} \tag{49}$$

Also,

$$\mu_m^{++}(1) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu_m^{**}(1) & m \geqslant 1. \end{cases} \tag{50}$$

(c) If $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, then

$$\mu_m^{**}(2) = \begin{cases} \infty & m = 0 \\ \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{1}{q^{(d_0-e_0)m} - 1} & m \geqslant 1. \end{cases} \tag{51}$$

Also

$$\mu_m^{++}(2) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu_m^{**}(2) & m \geqslant 1. \end{cases} \tag{52}$$

(3) $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$, then

$$\mu_m^{**}(3) = \begin{cases} \infty & m = 0 \\ \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{1}{q^{\frac{1}{2}(f_0-s_0)m} - 1} & m \geqslant 1. \end{cases} \tag{53}$$

Also,

$$\mu_m^{++}(3) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu_m^{**}(3) & m \geqslant 1. \end{cases} \tag{54}$$

Here the coefficients F and the symbol of summation $\sum_{(m)}$ and the parameters Ω_1, Ω_2 and M are as in theorem 1.

It is important to make the following observation. To get as much information as possible about the asymptotic distribution of zeros when the moments μ'_m of the conventional asymptotic density of zeros $\rho(x) = \lim_{N\to\infty} \rho_N(x)$ diverge, a normalization factor D is often used in theorems 2, 3 and 4, i.e. it is usually defined as an asymptotic density of zeros of the form

$$f(x) = \lim_{N\to\infty} C\rho_N(Dx) \tag{55}$$

where the factors C and D are chosen so that the moments μ_m of $f(x)$ given by

$$\mu_m = \lim_{N\to\infty} CD^m \mu_m'^{(N)} \tag{56}$$

are finite [40]. This is the great advantage of the densities of $f(x)$ type. The scaling factor D turns out to be a function of N and/or q^N. A detailed analysis of this procedure is carried out in section 5.

4. Determining the discrete density of zeros

Here theorem 1 will be proved. Let us consider the polynomial $P_N(x)_q$, N being a very large number, defined by the expressions (1)–(4), i.e. that

$$P_N(x) = (x - a_N)P_{N-1}(x) - b_{N-1}^2 P_{N-2}(x) \tag{57}$$

where a_N and b_N^2 are the values of a_n and b_n^2 given by equation (2) for $n = N$. First, let us find what are the N-dominant terms in the expressions (2) for a_N and b_{N-1}^2. Replacing n by N in equation (2) and taking into account that

$$\begin{aligned}
&\sum_{m=0}^{A}\left(\sum_{i=0}^{g_m} \alpha_i^{(m)} N^{g_m-i}\right) q^{d_m N} \sim \left(\sum_{i=0}^{g_0} \alpha_i^{(0)} N^{g_0-i}\right) q^{d_0 N} \sim \alpha_0^{(0)} N^{g_0} q^{d_0 N} \\
&\sum_{m=0}^{A'}\left(\sum_{i=0}^{h_m} \beta_i^{(m)} N^{h_m-i}\right) q^{e_m N} \sim \left(\sum_{i=0}^{h_0} \beta_i^{(0)} N^{h_0-i}\right) q^{e_0 N} \sim \beta_0^{(0)} N^{h_0} q^{e_0 N}
\end{aligned} \tag{58}$$

it is easy to obtain that

$$a_N \sim \frac{\alpha_0^{(0)}}{\beta_0^{(0)}} N^{g_0-h_0} q^{(e_0-d_0)N} \tag{59}$$

and, in a similar way, it is easy to obtain that

$$b_N^2 \sim \frac{\theta_0^{(0)}}{\gamma_0^{(0)}} N^{k_0-l_0} q^{(f_0-s_0)N}. \tag{60}$$

The symbol $\sim$ means, as already pointed out, *behaves with N as*. To get (58) the conditions (3) and (4) have been used. Remark that, taking into account equations (59), (60), equation (2) may be written as

$$\begin{aligned}
a_n &= \frac{\alpha_0^{(0)}}{\beta_0^{(0)}} n^{(g_0-h_0)} q^{(e_0-d_0)n} + \mathrm{O}(n^{g_0-h_0-1} q^{(e_0-d_0)n}) \\
b_n^2 &= \frac{\theta_0^{(0)}}{\gamma_0^{(0)}} n^{(k_0-l_0)} q^{(f_0-s_0)n} + \mathrm{O}(n^{k_0-l_0-1} q^{(f_0-s_0)n})
\end{aligned} \tag{61}$$

for $n \geqslant 1$. To calculate the discrete density of zeros $\rho_N(x)$ of the polynomial $P_N(x)_q$, it may first be assumed to be characterized by the knowledge of all its moments $\{\mu_m'^{(N)}, m = 0, 1, 2, \ldots, N\}$ defined by (5).

Taking the values (61) of a_n and b_n^2 into equation (7), we obtain for $\mu_m'^{(N)}$ the following values:

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \times \sum_{i=1}^{N-t} \left[\prod_{k=0}^{j-1} (i+k)^{(g_0-h_0)r_{k+1}'+(k_0-l_0)r_{k+1}}\right] (i+j)^{(g_0-h_0)r_{j+1}'} q^{\Omega_2+i\Omega_1}. \tag{62}$$

If we take into equation (62) the dominant term then it reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} q^{\Omega_2} \sum_{i=1}^{N-t} i^M q^{i\Omega_1} \tag{63}$$

with the following notations

$$R = \sum_{i=1}^{j} r_i \qquad R' = \sum_{i=1}^{j-1} r_i'$$

and

$$\Omega_1 = (d_0 - e_0)R' + (f_0 - s_0)R$$
$$\Omega_2 = (d_0 - e_0)\sum_{k=1}^{j} k r_{k+1}' + 2(f_0 - s_0)\sum_{k=1}^{j-1} k r_{k+1} \tag{64}$$
$$M = (g_0 - h_0)R' + (k_0 - l_0)R.$$

It should be noted that, because of relation (8), $R' + 2R = m$ and consequently the parameters Ω_1 and M may be written in the form

$$\Omega_1 = \left[(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)\right] R' + \frac{m}{2}(f_0 - s_0) \tag{65}$$

$$M = \left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right] R' + \frac{m}{2}(k_0 - l_0) \tag{66}$$

which are the expressions (22) and (24) given in the previous section.

To go further the i summation has to be performed in equation (63). In doing that two cases appear when expression (65) of Ω_1 is analysed:

(1) $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0$

(2) $d_0 - e_0 \neq 0$ and/or $\frac{1}{2}(f_0 - s_0) \neq 0$.

Let us see how equation (63) gets simplified in each case.

Case 1. $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0$.

In this case $\Omega_1 = \Omega_2 = 0$ and since

$$\sum_{i=1}^{N-t} i^M \sim (N-t)^{M+1} \qquad N \gg 1.$$

Equation (63) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{M+1}. \tag{67}$$

To further simplify this expression, we examine equation (66) of M. it is easy to find three different subcases corresponding to $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$ and $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, respectively. Let us study what happens for each subcase.

(1) (a) $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$. Note that

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{positive}} R' + \frac{m}{2}(k_0 - l_0).$$

Then the dominant term is obtained when $R' = m$ and $R = 0$, i.e. for the partition $(m, 0, 0, \ldots, 0)$. Therefore $M = (g_0 - h_0)m$ and expression (67) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(m, 0, 0, \ldots, 0) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m N^{(g_0-h_0)m+1}.$$

Since $F(m, 0, 0, \ldots, 0) = 1$ according to (9), it is clear that this relation is expression (11) of theorem 1.

(b) $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$. Then $M = \frac{m}{2}(k_0 - l_0)$ and (67) takes the form

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{\frac{m}{2}(k_0-l_0)+1}.$$

This expression coincides with (12) given in theorem 1.

(c) $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$. Note that

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0).$$

Then the dominant term is obtained when $2R = m$ and $R' = 0$, i.e. for the partition $(0, m, 0, \ldots, 0)$. Therefore $M = \frac{1}{2}(k_0 - l_0)$ and

$$\mu_m'^{(N)} \sim \sum_{(m)} F(0, m, 0, 0, \ldots, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} N^{\frac{1}{2}(k_0-l_0)+1}$$

which is the expression (13) given in theorem 1, since $F(0, m, 0, 0, \ldots, 0) = 1$.

Case 2. $d_0 - e_0 \neq 0$ and/or $\frac{1}{2}(f_0 - s_0) \neq 0$.

Here we are obliged to perform the i summation of (63). We have

$$\begin{aligned}\sum_{i=1}^{N-t} i^M q^{i\Omega_1} &= \frac{1}{(\ln q)^M} \sum_{i=1}^{N-t} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M} q^{i\Omega_1} \\ &= \frac{1}{(\ln q)^M} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M} \sum_{i=1}^{N-t} q^{i\Omega_1} \\ &= \frac{1}{(\ln q)^M} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M} \left[\frac{q^{\Omega_1} - q^{\Omega_1(N-t+1)}}{1 - q^{\Omega_1}}\right].\end{aligned}$$

Depending on whether q^{Ω_1} is smaller or bigger than unity, this summation has a N behaviour or another, indeed,

$$q^{\Omega_1} - q^{\Omega_1(N-t+1)} \sim \begin{cases} q^{\Omega_1} & \text{if } q^{\Omega_1} < 1 \\ -q^{\Omega_1(N-t+1)} & \text{if } q^{\Omega_1} > 1. \end{cases} \tag{68}$$

Then

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} \sim \frac{1}{(\ln q)^M} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M} \left[\frac{q^{\Omega_1}}{1 - q^{\Omega_1}}\right] \qquad \text{if } q^{\Omega_1} < 1 \tag{69}$$

and

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} \sim \left[\frac{q^{\Omega_1(N-t+1)}}{q^{\Omega_1}-1} N^M\right] \qquad \text{if } q^{\Omega_1} > 1. \tag{70}$$

Therefore, from (68) it is clear that to further reduce the expression (63) of the quantities $\mu_m'^{(N)}$ we have necessarily to distinguish the following two subcases: $q^{\Omega_1} < 1$ (i.e. $\Omega_1 < 0$) for all partitions of m and $q^{\Omega_1} > 1$ (i.e. $\Omega_1 > 0$) for at least one partition of m. Taking into account (65), these two subcases occur provided that

(1) (a) $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$

(b) $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$

(c) $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$

(2) $d_0 - e_0 > 0$ and/or $f_0 - s_0 > 0$

respectively. Let us see how the moments $\mu_m'^{(N)}$ given by (63) simplify in these two cases separately.

(a) Case 2a:

(i) $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$ in such a way that $\Omega_1 \neq 0$. The replacement of the i summation given by (69) in (63) leads to

$$\mu_m'^{(N)} \sim \sum_{(m)} \frac{F(r_1', r_1, \ldots, r_{j+1}')}{q^{-\Omega_2}(\ln q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{\mathrm{d}^M}{\mathrm{d}\Omega_1^M}\left(\frac{q^{\Omega_1}}{1-q^{\Omega_1}}\right)$$

which is expression (14) of theorem 1.

(ii) $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$ and $g_0 - h_0 = k_0 - l_0 = 0$. Since

$$M = \left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right] R' + \frac{m}{2}(k_0 - l_0) = 0$$

then

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} = \sum_{i=1}^{N-t} q^{i\Omega_1} = q^{\Omega_1}\left[\frac{1-q^{\Omega_1(N-t)}}{1-q^{\Omega_1}}\right]$$

where $\Omega_1 = (f_0 - s_0)R$ (see (64)). For $N \gg 1$ it is clear from the last expression that the i summation is a decreasing and convex upward function, which has a maximum when $\Omega_1 = 0$, i.e. when $R = 0$ and $R' = m$ and it is equal to N. This corresponds to all partitions $(r_1', 0, \ldots, 0, r_{j+1}')$. Note that (see (64))

$$\Omega_2 = \underbrace{(d_0 - e_0)}_{=0} \sum_{k=1}^{j} k r_{k+1}' + 2(f_0 - s_0) \sum_{k=1}^{j-1} k \underbrace{r_{k+1}}_{=0} = 0.$$

Then (63) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', 0, \ldots, 0, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} N$$

which coincides with expression (15) of theorem 1.

(iii) $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$ and $g_0 - h_0 = k_0 - l_0 = 0$. Here $\Omega_1 = (d_0 - e_0)R' \leqslant 0$. Then, as in the previous case, we have the conditions

$$\Omega_1 = 0 \qquad \Omega_2 = 0 \qquad i \text{ summation } = N$$

and (63) reduces as expression (16) of theorem 1.

(b) Case 2b: $d_0 - e_0 > 0$ and/or $f_0 - s_0 > 0$. Here from (70) and (63) one gets

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, \ldots, r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{q^{\Omega_2+(1-t)\Omega_1}}{q^{i\Omega_1}-1} q^{i\Omega_1 N} N^M. \tag{71}$$

To go further in the analysis of the N dependence of $\mu_m'^{(N)}$ one has to analyse expression (65) which defines Ω_1. A simple study allows us to distinguish the following three situations:

(i) $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$

(ii) $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$

(iii) $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$.

Now we shall examine the reduction of (71) in these situations.

(i) $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$. From (65) and (71) one easily finds that the dominant term in the m summation correspond to that for which $R' = m$, because

$$\Omega_1 = \underbrace{\left[(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)\right]}_{\text{positive}} R' + \frac{m}{2}(f_0 - s_0).$$

Then $R = 0$, $\Omega_1 = m(d_0 - e_0)$, $M = (g_0 - h_0)$, the corresponding partition is $(m, 0, \ldots, 0)$ and then $\Omega_2 = 0$ and $t = 0$. Therefore

$$\mu_m'^{(N)} \sim \sum_{(m)} F(m, 0, 0, \ldots, 0) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{m} \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m}.$$

Since $F(m, 0, 0, \ldots, 0) = 1$ according to (9) this relation is expression (17) of theorem 1.

(ii) $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$. Here we have $\Omega_1 = \frac{m}{2}(f_0 - s_0) = (d_0 - e_0)m$, that is a fixed number for all partitions of m. Then, in expression (71) we are obliged to study the parameter M given by (66) to know the N-dominant term of the m summation. The analysis of expression (66) leads us to separate the following three possibilities:

(A) $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$

(B) $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$

(C) $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$.

For the case $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$ the dominant term is the one corresponding to the condition when N^M is maximum. It occurs when $R' = m$, $R = 0$ because

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{positive}} R' + \frac{m}{2}(k_0 - l_0).$$

It corresponds to the partition $(m, 0, \ldots, 0)$, for which $F(m, 0, 0, \ldots, 0) = 1$, $t = 0$, $\Omega_2 = 0$, $M = (g_0 - h_0)m$. Then, equation (71) reduces as

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{m} \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m}$$

which coincides with equation (18) of theorem 1.

For the case $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$ it turns out that $M = (g_0 - h_0)m$, $\Omega_1 = (d_0 - e_0)$ and expression (71) easily transforms into (19) of theorem 1.

For the case $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$ we have, as before, $\Omega_1 = (d_0 - e_0)m$ and the dominant term is the one corresponding to the partition $(0, m, 0, \ldots, 0)$. It is because

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0).$$

Then, the maximum of N^M occurs for $R' = 0$, $R = \frac{m}{2}$. Therefore $t = 1$, $\Omega_2 = 0$, $M = \frac{1}{2}(k_0 - l_0)$ and (71) reduces as

$$\mu_m'^{(N)} \sim F(0, m, 0, \ldots, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{(d_0-e_0)mN}}{q^{(d_0-e_0)m} - 1} N^{\frac{1}{2}(k_0-l_0)m}$$

which is expression (20) of theorem 1 since $F(0, m, 0, \ldots, 0) = 1$.

(iii) $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$. Since

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0)$$

then the dominant term in the m summation of expression (71) is the one corresponding to the partition $(0, m, 0, \ldots, 0)$. Therefore $R' = 0$, $R = \frac{m}{2}$, $t = 1$, $\Omega_2 = 0$, $M = \frac{1}{2}(k_0 - l_0)$ and

$$\mu_m'^{(N)} \sim F(0, m, 0, \ldots, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{\frac{1}{2}(f_0-s_0)mN}}{q^{\frac{1}{2}(f_0-s_0)m} - 1} N^{\frac{1}{2}(f_0-s_0)m}$$

which coincides with (21) since $F(0, m, 0, \ldots, 0) = 1$.

This completely proves theorem 1. □

As a conclusion to this section we provide a scheme with all the different possibilities obtained in this section.

Scheme. The characterization of general q-polynomials by its spectral properties.

(1)

$$d_0 - e_0 = \tfrac{1}{2}(f_0 - s_0) \begin{cases} (a)\ g_0 - h_0 > \tfrac{1}{2}(k_0 - l_0) \\ (b)\ g_0 - h_0 = \tfrac{1}{2}(k_0 - l_0) \\ (c)\ g_0 - h_0 < \tfrac{1}{2}(k_0 - l_0) \end{cases}$$

(2)

$$\begin{matrix} d_0 - e_0 \neq 0 \\ f_0 - s_0 \neq 0 \end{matrix} \begin{cases} (a)\ \begin{matrix} d_0 - e_0 \leqslant 0 \\ f_0 - s_0 \leqslant 0 \end{matrix} \begin{cases} (i) \begin{cases} d_0 - e_0 < 0 \\ f_0 - s_0 < 0 \end{cases} & \Omega_1 \neq 0 \\ (ii) \begin{cases} d_0 - e_0 = 0 \\ f_0 - s_0 < 0 \end{cases} & g_0 - h_0 = k_0 - l_0 = 0 \\ (iii) \begin{cases} d_0 - e_0 < 0 \\ f_0 - s_0 = 0 \end{cases} & g_0 - h_0 = k_0 - l_0 = 0 \end{cases} \\ (b)\ \begin{matrix} d_0 - e_0 > 0 \\ \text{and/or} \\ f_0 - s_0 > 0 \end{matrix} \begin{cases} (i)\ d_0 - e_0 > \tfrac{1}{2}(f_0 - s_0) \\ (ii)\ d_0 - e_0 = \tfrac{1}{2}(f_0 - s_0) \begin{cases} (A)\ g_0 - h_0 > \tfrac{1}{2}(k_0 - l_0) \\ (B)\ g_0 - h_0 = \tfrac{1}{2}(k_0 - l_0) \\ (C)\ g_0 - h_0 < \tfrac{1}{2}(k_0 - l_0) \end{cases} \\ (iii)\ d_0 - e_0 < \tfrac{1}{2}(f_0 - s_0) \end{cases} \end{cases}$$

5. Searching for a normalized density of zeros

In this section the asymptotic distribution of zeros of the polynomial $P_N(x)_q$ defined by equations (1)–(4) will be discussed. In particular theorems 2–4 will be proved. The starting point will be theorem 1.

From theorem 1, we observe that the moments $\mu_m'^{(N)}$ of the (non-normalized) density of zeros $\rho_N(x)$ depends on N as follows:

$$\begin{array}{ll} N^{am+1} & \text{in case 1} \\ \text{Constant} & \text{in subcase 2ai} \\ N & \text{in subcases 2aii–2aiii} \\ N^{am}q^{bmN} & \text{in case 2b} \end{array} \tag{72}$$

where the constants a and b are known and distinct for each case. Obviously we would like to have a normalized density of zeros $\rho_N^{\text{norm}}(x)$. The usual way to have it is to impose that the *moment of order zero* be equal to unity, which allows us to write

$$\rho_N^{\text{norm}}(x) = \frac{1}{N}\rho_N(x) \tag{73}$$

whose moments $\tilde{\mu}_m'^{(N)}$ will be related to those of $\rho_N(x)$ by

$$\tilde{\mu}_m'^{(N)} = \frac{1}{N}\mu_m'^{(N)} \qquad m \geqslant 0. \tag{74}$$

Then, from (72) and (74) it is clear that the N dependence of the moments of the *normalized-to-unity* density of zeros is given by

$$\begin{array}{ll} N^{am} & \text{in case 1} \\ N^{-1} & \text{in subcase 2ai} \\ \text{Constant} & \text{in subcases 2aii–2aiii} \\ N^{am-1}q^{bmN} & \text{in case 2b.} \end{array} \tag{75}$$

As said before, we are interested in the asymptotic density of zeros. If this is defined by

$$\rho(x) = \lim_{N\to\infty} \rho_N(x) \tag{76}$$

then taking into account that $\mu_m'^{(N)}$ have a N dependence of the form (72), its moments μ_m' given by

$$\mu_m' = \lim_{N\to\infty} \mu_m'^{(N)}$$

will be infinity in case 1, subcases 2aii and 2aiii and in case 2b; and constant given by (14) in subcase 2ai. Therefore, the expressions (27), (33), (35), (37) of theorems 2–4, respectively, have been proved.

If some information is required about the asymptotic distribution of zeros in case 1, subcases 2aii and 2aiii and in case 2b, a normalization factor and/or a scaling factor needs to be introduced into the density $\rho_N(x)$ in the sense discussed in equations (55) and (56). Let us first think of a *scaled* density. For case 1 there is no scaling factor D which leads to an asymptotic density of zeros whose moments have non-zero finite values unless the scaling factor be of the form $D = N^{-a-\frac{1}{m}}$ but this is not useful since it would need a definition of a different *scaled* asymptotic density function for each moment. Contrary to this, for case 2b we can consider the scaling factor $D = N^{-a}q^{-bN}$ and define the discrete density of zeros given by

$$\rho_N^{**}(x) = \rho_N\left(\frac{x}{q^{bN}N^a}\right)$$

and the asymptotic density of zeros given by

$$\rho^{**}(x) = \lim_{N\to\infty} \rho_N\left(\frac{x}{q^{bN}N^a}\right) \tag{77}$$

when moments μ_m^{**} are according to (56), as follows

$$\mu_m^{**} = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{q^{mbN}N^{am}}. \tag{78}$$

From (72) and (78), it is clear that all the quantities μ_m^{**} have finite values. The only omission is the value of parameters a and b for the different subcases of case 2b.

For subcases 2bi, 2biiA and 2biiB it turns out that $a = g_0 - h_0$ and $b = d_0 - e_0$. Then, as in expression (77), we can define the asymptotic density function $\rho_1^{**}(x)$ in the form

$$\rho_1^{**}(x) = \lim_{N\to\infty} \rho_N\left(\frac{xq^{-(d_0-e_0)N}}{N^{(g_0-h_0)}}\right) \tag{79}$$

whose moments, $\mu_m^{**}(1)$, given by

$$\mu_m^{**}(1) = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(g_0-h_0)m}q^{(d_0-e_0)mN}} \tag{80}$$

have, according to (17) and (18), the values (for $m \geqslant 1$)

$$\mu_m^{**}(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(d_0-e_0)}}{q^{m(d_0-e_0)}-1} \tag{81}$$

in subcases 2bi and 2biiA, and, according to (19), the values

$$\mu_m^{**}(1) = \sum_{(m)} F(r_1', r_1, \ldots, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^R \frac{q^{\Omega_2+m(1-t)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} \tag{82}$$

in subcase 2biiB. Note that expressions (81) and (82) are identical to (47) and (49) of theorem 4, respectively. Similarly, for subcases 2biiC it turns out that $a = \frac{1}{2}(k_0 - l_0)$ and $b = d_0 - e_0$. Then, as in expression (77), we define the asymptotic density function $\rho_2^{**}(x)$ by (40), whose moments $\mu_m^{**}(2)$ given by (43) have, according to (20), the values given by (51). Finally, for subcase 2biii we have the density $\rho_3^{**}(x)$ defined by (40), whose moments $\mu_m^{**}(3)$, given by (43), have, according to (21), the values given by (53). For the entire case 2b it happens that, according to (78) and since $\mu_0'^{(N)} = N$,

$$\mu_0^{**} = \mu_0^{**}(1) = \mu_0^{**}(2) = \mu_0^{**}(3) = \infty$$

as in theorem 4 is also pointed out.

Let us now search for a *normalized-to-unity* asymptotic density of zeros. The simplest way is to define it as

$$\rho_1(x) = \lim_{N\to\infty} \rho_N^{\text{norm}}(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N(x) \tag{83}$$

where equation (73) has been used. Its moments given by

$$\mu_0'(1) = 1 \qquad \mu_m'(1) = \lim_{N\to\infty} \frac{1}{N}\mu_m'^{(N)} \qquad m \geqslant 1 \tag{84}$$

have, taking into account (75), the following values

$$\mu'_0(1) = 1$$

$$\mu'_m(1) = \begin{cases} \infty \qquad m \geqslant 1 & \text{in cases 1 and 2b} \\ 0 \qquad m \geqslant 1 & \text{subcase 2ai} \\ \sum_{(m)} F(r'_1, 0, \ldots, 0, r'_{j+1}) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \qquad m \geqslant 1 & \text{subcase 2aii} \\ \sum_{(m)} F(0, r_1, 0, \ldots, r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \qquad m \geqslant 1 & \text{subcase 2aiii.} \end{cases} \tag{85}$$

Then expressions (34)–(38) of theorem 3 have been demonstrated. So, theorem 3 has been entirely proved.

For cases 1 and 2b one would like to have information more useful than that expressed by (85), keeping the *normalization to unit* of the density $\rho_1(x)$ given by (83). Therefore, the spectrum of zeros has to be *compressed* by introducing a scaling factor. In case 1 it is very easy to find that factor by looking at expression (75): it is $D = N^{-a}$. Then, from (75) and (83), the density function is defined as

$$\rho^*(x) = \lim_{N\to\infty} \rho_N^{\text{norm}}\left(\frac{x}{N^a}\right) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^a}\right) \tag{86}$$

whose moments are, according to (56) and (84), as

$$\mu_0^* = 1 \qquad \mu_m^* = \lim_{N\to\infty} \frac{\mu_m^{\prime(N)}}{N^{am+1}} \qquad m \geqslant 1. \tag{87}$$

From (72) and (87) it is obvious that the quantities μ_m^* have finite values. We have only to take the values of a in the different subcases of case 1. For subcase 1a, $a = g_0 - h_0$; then here it is convenient to define, according to (86), the following asymptotic density of zeros

$$\rho_1^*(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{g_0-h_0}}\right)$$

whose moments are, according to (87) and (11), as follows

$$\mu_0^*(1) = 1 \qquad \mu_m^*(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \qquad m \geqslant 1$$

which is expression (28) of theorem 2.

For subcases 1b and 1c, it turns out that $a = \frac{1}{2}(k_0 - l_0)$, which defines the following asymptotic density of zeros

$$\rho_2^*(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{\frac{1}{2}(k_0-l_0)}}\right)$$

whose moments have, according to (87) and (12), the values ($\mu_0^*(2) = 1$)

$$\mu_m^*(2) = \sum_{(m)} F(r'_1, r_1, \ldots, r'_{j+1}) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \qquad m \geqslant 1$$

for subcase 1b, and, according to (87) and (13), the values

$$\mu_0^*(2) = 1 \qquad \mu_m^*(2) = \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \qquad m \geqslant 1$$

for subcase 1c. Remarks that the last two expressions coincide with expressions (29) and (30) of theorem 2, respectively. Then this theorem has been entirely proved. □

For case 2b the scaled *normalization-to-unity* asymptotic density function of form (86) would also have all its moments of order other than zero equal to infinite. No other scaling factor would be able to make finite these moments unless $D = N^{-a+\frac{1}{m}} q^{mN}$, but this factor is of usefulness for reasons already discussed. Therefore, we are obliged to change the normalization factor in this subcase. Here the discrete density of zeros $\rho_N^+(x)$ is normalized so that its moments are defined by

$$\mu_m^{+(N)} = \frac{q^m - 1}{q^{mN} - 1} \mu_m'^{(N)} \qquad m \geqslant 0$$

i.e. that

$$\rho_N^+(x) = \frac{(m)_q}{(mN)_q} \rho_N(x) \tag{88}$$

when $(m)_q$ and $(mN)_q$ are q numbers defined by equation (45). This normalization factor has the following relevant property: it tends to N^{-1} if $m \to 0$ and $q \to 1$. In particular, this implies that

$$\mu_0^{+(N)} = 1.$$

Furthermore, for case 2b under consideration it turns out that the N dependence of $\mu_m^{+(N)}$ is as $N^{am} q^{(b-1)mN}$. This dependence suggests the analysis of the asymptotic spectrum of zeros by means of the asymptotic density function defined by

$$\rho^{++}(x) = \lim_{N\to\infty} \rho_N \left(\frac{x}{N^a q^{(b-1)N}} \right) \tag{89}$$

whose moments, μ_m^{++}, are given by

$$\mu_m^{++} = \lim_{N\to\infty} \frac{\mu_m^{+(N)}}{N^{am} q^{(b-1)mN}} = \lim_{N\to\infty} \frac{(q^m - 1)\mu_m'^{(N)}}{(q^{mN} - 1) N^{am} q^{(b-1)mN}}. \tag{90}$$

Taking into account this expression together with values (17)–(21) of $\mu_m'^{(N)}$ given in theorem 1, one observes that for subcases 2bi, 2biiA and 2biiB parameters a and b take the values

$$a = g_0 - h_0 \qquad b = d_0 - e_0$$

and the appropriate asymptotic density of zeros is, according to (88), (89), the function $\rho_1^{++}(x)$ given by (41) in theorem 4.

For subcase 2biiC it turns out that

$$a = \tfrac{1}{2}(k_0 - l_0) \qquad b = d_0 - e_0.$$

Then, the appropriate asymptotic density of zeros for this subcase is, according to (88), (89), the function $\rho_2^{++}(x)$ given by (41) in theorem 4.

Finally for subcase 2biii $a = \frac{1}{2}(k_0 - l_0)$, $b = \frac{1}{2}(f_0 - s_0)$ and the appropriate asymptotic density of zeros is, according to (88), (89), the function $\rho_3^{++}(x)$ given by (41) in theorem 4.

Now equation (90) and values (17)–(21) for $\mu_m'^{(N)}$ gives, in a straightforward manner, the moments $\mu_m^{++}(1)$, $\mu_m^{++}(2)$ and $\mu_m^{++}(3)$ of the asymptotic density functions $\rho_1^{++}(x)$, $\rho_2^{++}(x)$ and $\rho_3^{++}(x)$. Indeed, the values of these quantities are given by equations (48) for subcases 2bi and 2biiA, (50) for subcase 2biiB, (52) for subcase 2biiC and (54) for subcase 2biii, respectively. This entirely proves theorems 2–4. □

6. Applications

In this section we will use the theorems obtained in the two previous sections to investigate the spectral properties of several known families of orthogonal q-polynomials. Let us note that for a finite polynomial sequence (e.g. Hahn, Racah and Kravchuk polynomials), i.e. when the degree n of the polynomial is bounded by a fixed parameter N (not to be confused with the same letter previously used as the generic degree of polynomials), it is assumed that N is sufficiently large and $1 \ll n \leqslant N$ so that equation (61) can be fulfilled.

6.1. The q-Hahn polynomials $h_n^{\alpha,\beta}(q^{-x}, N)$

The q-Hahn polynomials $h_n^{\alpha,\beta}(q^{-x}, N)$ play a fundamental role in the representation theory of the q-algebras $SU_q(2)$ and $SU_q(1,1)$ (see [20, 22, 21]). They also appear in numerous physical applications since, for example, the Clebsch–Gordan coefficients of the q-algebras $SU_q(2)$ and $SU_q(1,1)$ are proportional to them. The theory and applications of the q-Hahn and classical Hahn polynomials have some close parallels. So, for example, q-Hahn and classical Hahn polynomials appear in the analysis of functions on the lattice of subspaces of a finite vector space and the lattice of subsets of a finite set, respectively. These polynomials verify the recurrence relation [3, p 59]

$$h_n^{\alpha,\beta}(q^{-x}, N) = [q^{-x} - (1 - A_{n-1} - C_{n-1})]h_{n-1}^{\alpha,\beta}(q^{-x}, N) + B_{n-1}h_{n-2}^{\alpha,\beta}(q^{-x}, N) \tag{91}$$

where $B_n = A_{n-1}C_n$, and the A and C parameters are

$$A_n = \frac{(1-\alpha q^{1+n})(1-\alpha\beta q^{1+n})(1-q^{-N+n})}{(1-\alpha\beta q^{1+2n})(1-\alpha\beta q^{2+2n})}$$
$$C_n = -\frac{\alpha q^n(1-q^n)(1-\beta q^n)(q^{-N}-\alpha\beta q^{1+n})}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{1+2n})}.$$

Let us also point out that

$$\kappa_{n+1}A_n = \kappa_n \tag{92}$$

where κ_n is the leading coefficient of the polynomial. The comparison of equations (91) and (1) gives that

$$a_n^{\text{num}} \sim \alpha_0^{(0)} q^{3n} = \alpha^2\beta(1+\beta)q^{N+1}q^{3n} \qquad a_n^{\text{den}} \sim \beta_0^{(0)} q^{4n} = \alpha^2\beta^2 q^N q^{4n}$$

and

$$(b_n^{\text{num}})^2 \sim \theta_0^{(0)} q^{7n} = \alpha^4\beta^3 q^{-N} q^{7n} \qquad (b_n^{\text{den}})^2 \sim \gamma_0^{(0)} q^{8n} = \alpha^4\beta^4 q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \dots N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 7 \qquad s_0 = 8.$$

This is the case $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$, i.e. subcase 2ai. Therefore, equations (33) and (34) of theorem 3 give us the moments

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, \dots, r_j, r'_{j+1}) q^{-\sum_{k=1}^{j} k r'_{k+1} - 2\sum_{k=1}^{j-1} k} \left[\frac{q(1+\beta)}{\beta}\right]^{R'} \left[\frac{\alpha}{q^N(q+q^{-1})}\right]^{R} \frac{1}{q^{\frac{m}{2}-1}} \tag{93}$$

for the asymptotic density of zeros $\rho(x)$ defined by equation (31), and

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geqslant 1 \end{cases} \tag{94}$$

for the corresponding asymptotic quantity $\rho_1(x)$ given by equation (31).

6.2. q-Kravchuk polynomials $k_n^p(q^{-x}, N)$

The matrix elements of the representations T^l of the $U_q(sl_2)$ quantum algebra are proportional to the q-Kravchuk polynomials (see [21, vol III, p 64]). According to [3, p 76] and taking into account equation (92) the three-term recurrence relation of these polynomials can be expressed as

$$k_n^p(q^{-x}, N) = [q^{-x} - (1 - A_{n-1} - C_{n-1})]k_{n-1}^p(q^{-x}, N) + B_{n-1}k_{n-2}^p(q^{-x}, N) \tag{95}$$

where $B_{n-1} = A_{n-1}C_{n-1}$, and

$$A_n = \frac{(1 + pq^n)(1 - q^{-K+n})}{(1 + pq^{2n})(1 + pq^{1+2n})}$$
$$C_n = -\frac{pq^{-1-K+2n}(1 - q^n)(1 + pq^{K+n})}{(1 + pq^{2n})(1 + pq^{-1+2n})}.$$

The comparison with (1) gives that

$$a_n^{\text{num}} \sim \alpha_0^{(0)} q^{3n} = pq(pq^N - 1)q^{3n} \qquad a_n^{\text{den}} \sim \beta_0^{(0)} q^{4n} = p^2 q^N q^{4n}$$

and

$$(b_n^{\text{num}})^2 \sim \theta_0^{(0)} q^{6n} = p^3 q^{-2} q^{6n} \qquad (b_n^{\text{den}})^2 \sim \gamma_0^{(0)} q^{8n} = q^{-3} p^4 q^N q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \ldots N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 6 \qquad s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = -2 < 0$, i.e. subcase 2ai. Therefore, equation (34) of theorem 3 gives us the values

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geqslant 1 \end{cases} \tag{96}$$

for the moments of the asymptotic density of zeros $\rho_1(x)$. Furthermore, since $\Omega_1 = \frac{1}{2}(f_0 - s_0) = -m$, $\Omega_2 = -(\sum_{k=1}^{j} kr'_{k+1} - 4\sum_{k=1}^{j-1} kr_{k+1})$ and $M = 0$, equation (33) of theorem 3 gives us

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, \ldots, r'_{j+1}) q^{-\Omega_2} \left[\frac{q(pq^N - 1)}{pq^N}\right]^{R'} \left[\frac{q^{1-N}}{p}\right]^{R} \frac{1}{q^m - 1} \qquad m \geqslant 0 \tag{97}$$

for the moments of the spectral quantity $\rho(x)$ defined by equation (31).

6.3. q-Racah polynomials $R_n(\mu(x), \alpha, \beta, \gamma, \delta)$. $u(x) = q^{-x} + \gamma\delta q^{x+1}$

The important role that the $6j$ symbols play in the quantum angular momentum theory is well known (see [6]). It is known that the q-analogue of the Racah coefficients ($6j$ symbols) for the q-algebra $U_q(sl_2)$ are proportional to the q-Racah polynomials (see [21, vol III, p 70]). From the three-term recurrence relation of these polynomials [3, p 53], as well as equation (92), we can rewrite [3, equation 3.15.3] in the form

$$\mu(x)R_{n-1}(\mu(x), \alpha, \beta, \gamma, \delta) = R_n(\mu(x), \alpha, \beta, \gamma, \delta) + [1 + \gamma\delta q - (1 - A_{n-1} - C_{n-1})] \\ \times R_{n-1}(\mu(x), \alpha, \beta, \gamma, \delta) + B_{n-1}R_{n-2}(\mu(x), \alpha, \beta, \gamma, \delta) \tag{98}$$

where $B_{n-1} = A_{n-1}C_{n-1}$, and

$$A_n = \frac{(1-\alpha q^{1+n})(1-\alpha\beta q^{1+n})(1-\beta\delta q^{1+n})(1-\gamma q^{1+n})}{(1-\alpha\beta q^{1+2n})(1-\alpha\beta q^{2+2n})}$$
$$C_n = \frac{q(1-q^n)(\delta-\alpha q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{1+2n})}.$$

The comparison with (1) gives that

$$a_n^{\text{num}} \sim -\alpha_0^{(0)} q^{3n} = q\alpha\beta(\alpha+\gamma+\alpha\beta+\beta\delta+\alpha\beta\delta+\alpha\gamma+\delta\gamma+\beta\gamma\delta)q^{3n}$$
$$a_n^{\text{den}} \sim -\beta_0^{(0)} q^{4n} = \alpha^2\beta^2 q^{4n}$$

and

$$(b_n^{\text{num}})^2 \sim \theta_0^{(0)} q^{8n} = q\alpha^4\beta^4\delta\gamma q^{8n} \qquad (b_n^{\text{den}})^2 \sim \gamma_0^{(0)} q^{8n} = \alpha^4\beta^4 q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \ldots N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 8 \qquad s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = 0$, i.e. subcase 2aiii. Therefore, equation (16) of theorem 3 yield the moments

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ \sum_{(m)} F(0, r_1, 0, \ldots, r_j, 0)[q\delta\gamma]^R & m \geqslant 1 \end{cases} \tag{99}$$

for the asymptotic densities of zeros $\rho_1(x)$ defined by equation (31).

6.4. *q-Askey and Wilson polynomials* $p_n(x, a, b, c, d)$

According to [3, p 51] and equation (92), the three-term recurrence relation for the q-Askey and Wilson polynomials can be rewritten as

$$x p_{n-1}(x, a, b, c, d) = p_n(x, a, b, c, d) + \tfrac{1}{2}[a + a^{-1} - (A_{n-1} + C_{n-1})] p_{n-1}(x, a, b, c, d) + B_{n-1} p_{n-2}(x, a, b, c, d) \tag{100}$$

where $B_{n-1} = A_{n-1}C_{n-1}$, and

$$A_n = \frac{(1-abcdq^{-1+n})(1-abq^n)(1-acq^n)(1-adq^n)}{a(1-abcdq^{2n})(1-abcdq^{-1+2n})}$$
$$C_n = \frac{a(1-bcq^{-1+n})(1-bdq^{-1+n})(1-cdq^{-1+n})(1-q^n)}{(1-abcdq^{-2+2n})(1-abcdq^{-1+2n})}.$$

The comparison with (1) gives

$$a_n^{\text{num}} \sim -\alpha_0^{(0)} q^{3n} = qabcd(abc+abd+acd+bcd+q(a+b+c+d))q^{3n}$$
$$a_n^{\text{den}} \sim -\beta_0^{(0)} q^{4n} = 2a^2b^2c^2d^2 q^{4n}$$

and

$$(b_n^{\text{num}})^2 \sim \theta_0^{(0)} q^{8n} = a^4b^4c^4d^4 q^{8n} \qquad (b_n^{\text{den}})^2 \sim \gamma_0^{(0)} q^{nn} = a^4b^4c^4d^4 q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \ldots N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 8 \qquad s_0 = 8.$$

This is the case $d_0-e_0=-1<0$ and $f_0-s_0=0$, i.e. subcase 2aiii. Therefore, equation (38) of theorem 3 gives us the moments

$$\mu'_m(1)=\begin{cases}1 & m=0\\ \sum_{(m)} F(0,r_1,0,\ldots,r_j,0) & m\geqslant 1.\end{cases} \tag{101}$$

for the asymptotic density of zeros $\rho_1(x)$ defined by equation (31).

6.5. Al Salam–Carlitz polynomials $u_n^\mu(x)$ and $v_n^\mu(x)$

In dealing with the q-harmonic oscillator, Askey and Suslov [16] have introduced the q-polynomials

$$u_n^\mu(x)=\mu^{-n}q^{-\frac{n(n-1)}{2}}U_n^{(-\mu)}(x).$$

where $U_n^{-\mu}(x)$ are the so-called Al Salam–Carlitz polynomials. These polynomials satisfy the recurrence relation [16]

$$xu_{n-1}^\mu(x)=u_n^\mu(x)+(1-\mu)q^{n-1}u_{n-1}^\mu(x)+\mu q^{n-2}(1-q^{n-1})u_{n-2}^\mu(x) \tag{102}$$

which is of the type (1) with the coefficients

$$a_n^{\rm num}=-\alpha_0^{(0)}q^n=(1-\mu)q^{-1}q^n \qquad a_n^{\rm den}=1$$

and

$$(b_n^{\rm num})^2\sim\theta_0^{(0)}q^{2n}=\mu q^{-1}q^{2n} \qquad (b_n^{\rm den})^2=\gamma_0^{(0)}q^{s_0 n}=1.$$

Then, $g_m=h_m=k_m=l_m=0$ for all $m=0,1,\ldots N$ and

$$d_0=1 \qquad e_0=0 \qquad f_0=2 \qquad s_0=0.$$

This is the case $d_0-e_0=1$ and $f_0-s_0=2$, i.e. subcase 2biiB. Therefore, equations (49) and (50) of theorem 4 give us the moments

$$\mu_m^{**}(1)=\begin{cases}\infty & m=0\\ \sum_{(m)} F(r_1',r_1,\ldots,r_{j+1}')[1-\mu]^{R'}\mu^R\dfrac{q^{\Omega_2}}{q^m-1} & m\geqslant 1\end{cases} \tag{103}$$

and

$$\mu_m^{++}(1)=\begin{cases}1 & m=0\\ \sum_{(m)} F(r_1',r_1,\ldots,r_{j+1}')[1-\mu]^{R'}\mu^R q^{\Omega_2} & m\geqslant 1\end{cases} \tag{104}$$

(where $\Omega_2=\sum_{k=1}^{j}kr_{k+1}'+4\sum_{k=1}^{j-1}kr_{k+1}-mt$) corresponding to the asymptotic quantities $\rho_1^{**}(x)$ and $\rho_1^{++}(x)$, respectively.

It has been encountered [15] that another class of the Al Salam–Carlitz polynomials, to be denoted by $v_n^\mu(x)$, is related also to the q-oscillator. So, it seems natural to search for its distribution of zeros. These polynomials satisfy the relation [15]

$$xv_{n-1}^\mu(x)=v_n^\mu(x)+(q+\mu)q^{-n-2}v_{n-1}^\mu(x)+\mu q^{-n-3}(q^{-n-1}-1)v_{n-2}^\mu(x). \tag{105}$$

Therefore, $d_0=-1$, $e_0=0$, $f_0=-1$, $s_0=0$. This corresponds to subcase 2ai. Then, equation (34) of theorem 3 gives us the moments

$$\mu'_m(1)=\begin{cases}1 & m=0\\ 0 & m\geqslant 1\end{cases} \tag{106}$$

for the asymptotic density $\rho(x)$. Furthermore, since $\Omega_1 = -\frac{1}{2}(R' + m)$, $\Omega_2 = -(\sum_{k=1}^{j} kr'_{k+1} - 2\sum_{k=1}^{j-1} kr_{k+1})$ and $M = 0$, equation (33) of theorem 3 gives

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, \ldots, r'_{j+1}) q^{-\Omega_2} [q^{-1}(q+\mu)]^{R'} \mu^R \frac{q^{-m}}{q^{\frac{1}{2}(R'+m)} - 1} \tag{107}$$

for the moments of the normalized-to-$\frac{1}{N}$ spectral quantity $\rho_1(x)$ defined by equation (31).

6.6. The little q-Jacobi polynomials $p_n(x, a, b)$

The little q-Jacobi polynomials $p_n(x, a, b)$ play a fundamental role (see e.g. [20]) in the representation theory of the q-algebra $U_q(sl_2)$ because they are the matrix elements of the representations T^l (see [21, vol III, p 51]). They satisfy the three-term recurrence relation [3, p 59]

$$p_n(x, a, b) = [x + A_{n-1} + C_{n-1}] p_{n-1}(x, a, b) + B_{n-1} p_{n-2}(x, a, b) \tag{108}$$

where A and C parameters are given by

$$A_n = \frac{q^n(1 - aq^{1+n})(1 - abq^{1+n})}{(1 - abq^{1+2n})(1 - abq^{2+2n})}$$
$$C_n = \frac{aq^n(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{1+2n})}$$

and $B_n = A_{n-1}C_n$. This relation is of the type (1) with the coefficients

$$a_n^{\text{num}} = \alpha_0^{(0)} q^{3n} = -ab(1+a)q^{3n} \qquad a_n^{\text{den}} = \beta_0^{(0)} q^{4n} = a^4 b^4 q^{4n}$$

and

$$(b_n^{\text{num}})^2 = \theta_0^{(0)} q^{6n} = a^3 b^2 q^{6n} \qquad (b_n^{\text{den}})^2 = \gamma_0^{(0)} q^{8n} = qa^4 b^4 q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \ldots, N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 6 \qquad s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = -2 < 0$, i.e. subcase 2ai. Therefore, equation (34) of theorem 3 gives us the moments

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geqslant 1 \end{cases} \tag{109}$$

for the asymptotic density of zeros $\rho(x)$. Furthermore, since $\Omega_1 = \frac{1}{2}(f_0 - s_0) = -m$, $\Omega_2 = -(\sum_{k=1}^{j} kr'_{k+1} - 4\sum_{k=1}^{j-1} kr_{k+1})$ and $M = 0$, equation (33) of theorem 3 gives us

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, \ldots, r'_{j+1}) q^{-\Omega_2} \left[\frac{1+a}{a}\right]^{R'} \left[\frac{-1}{aq}\right]^{R} \frac{1}{(q^m - 1)b^m} \tag{110}$$

which are the moments of asymptotic spectral quantity $\rho(x)$.

6.7. The big q-Jacobi polynomials $P_n(x, a, b, c)$

The big q-Jacobi polynomials $P_n(x, a, b, c)$ are defined in [3, p 57]. By use of the parameters

$$A_n = \frac{(1 - aq^{1+n})(1 - abq^{1+n})(1 - cq^{1+n})}{(1 - abq^{1+2n})(1 - abq^{2+2n})}$$
$$C_n = -\frac{acq^{1+n}(1 - q^n)(1 - bq^n)\left(1 - \frac{abq^n}{c}\right)}{(1 - abq^{2n})(1 - abq^{1+2n})}$$

we can rewrite its three-term recurrence relation [3, p 59] in the form

$$P_n(x, a, b, c) = [x + 1 - A_{n-1} - C_{n-1}]P_{n-1}(x, a, b, c) + B_{n-1}P_{n-2}(x, a, b, c) \qquad (111)$$

where $B_n = A_{n-1}C_n$. The comparison with (1) gives

$$a_n^{\text{num}} \sim \alpha_0^{(0)} q^{3n} = -qab(b + 1)(a + c)q^{3n} \qquad a_n^{\text{den}} \sim \beta_0^{(0)} q^{4n} = -a^2b^2q^{4n}$$

and

$$(b_n^{\text{num}})^2 \sim \theta_0^{(0)} q^{7n} = a^4b^3cqq^{7n} \qquad (b_n^{\text{den}})^2 \sim \gamma_0^{(0)} q^{6n} = a^4b^4q^{-1}q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, \ldots, N$ and

$$d_0 = 3 \qquad e_0 = 4 \qquad f_0 = 7 \qquad s_0 = 8.$$

This is the case $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$, i.e. subcase 2ai. Therefore, equations (33) and (34) of theorem 3 give us the moments

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, \ldots, r_j, r'_{j+1}) q^{-\sum_{k=1}^{j} kr'_{k+1} - 2\sum_{k=1}^{j-1} k} \times \left[\frac{(b + 1)(a + c)}{ab}\right]^{R'} \left[\frac{c}{b}\right]^{R} \frac{1}{q^{-R} - q^{-m}} \qquad (112)$$

for the asymptotic density of zeros $\rho(x)$ defined by equation (31), and

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geqslant 1 \end{cases} \qquad (113)$$

for the corresponding asymptotic quantity $\rho_1(x)$ given by equation (31).

6.8. The q-dual Hahn polynomials in the lattice $x(s) = [s]_q[s + 1]_q$

In this section we provide the asymptotic behaviour of the moments of zeros of the q-dual Hahn polynomials $W_n^{(c)}(x(s), a, b)_q$. These polynomials are connected with the Clebsh–Gordan of the q-algebras $SU_q(2)$ and $SU_q(1, 1)$ [24]. Using the above formulae we find the following asymptotic values of the moments $\mu_m'^{(N)} (m \geqslant 1)$

$$\mu_m'^{(N)} \sim \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1} + r'_i + r_i - 1)!}{(r_{i-1} - 1)! r_i! r'_i!} \frac{q^{5m-3R}}{[q^{-2c} - 1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q - q^{-1})^{2m}} \frac{q^{8m(N-t)}}{q^{4m} - 1}.$$

Using the normalized density of zeros

$$\rho_1^{++}(x) = \lim_{N\to\infty} \frac{q^m - 1}{q^{mN} - 1} \rho_N(xq^{-7N})$$

then, the corresponding moments are given by the expression for $m \geqslant 1$

$$\mu_0^{++}(1) = 1$$

$$\mu_m^{++}(1) = \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1} + r_i' + r_i - 1)!}{(r_{i-1} - 1)! r_i! r_i'!} \frac{q^{5m-3R}}{[q^{-2c} - 1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q - q^{-1})^{2m}} \frac{q^m - 1}{q^{4m} - 1}.$$

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